Lecture 13: October 9

Our next goal is to prove Theorem 10.2. From a polarized variation of Hodge structure of weight n on the punctured disk Δ^*, we constructed a period mapping $\Phi\colon \tilde{\mathbb{H}} \to D$, and a holomorphic mapping $\Psi\colon \Delta^* \to \check{D}$; the two are related by the formula $\Psi(e^z) = e^{-zR}\Phi(z)$.

Theorem. *With notation as above, the mapping*

$$\Delta^* \to D, \quad t \mapsto e^{-\frac{1}{2}R_N} e^{-\frac{1}{2}L(t)R_S} e^{\frac{1}{2}\log L(t)H}\Psi(t),$$

extends continuously over the origin in Δ. More precisely, the following is true:

(a) The limit

$$\hat{F}_H = \lim_{t\to 0} e^{-\frac{1}{2}L(t)R_S} e^{\frac{1}{2}\log L(t)H}\Psi(t) \in \check{D}$$

exists, and $e^{-\frac{1}{2}R_N}\hat{F}_H \in D$. In other words, $e^{-\frac{1}{2}R_N}\hat{F}_H$ defines a Hodge structure of weight n on the vector space V, polarized by the pairing h.

(b) The filtration $\hat{F}_H$ is compatible with the semisimple operators R_S and H, in the sense that $R_S\hat{F}_H^p \subseteq \hat{F}_H^p$ and $H\hat{F}_H^p \subseteq \hat{F}_H^p$ for every $p \in \mathbb{Z}$.

(c) One has $R_N\hat{F}_H^p \subseteq \hat{F}_H^{p-1}$ for every $p \in \mathbb{Z}$.

Unfortunately, the proof I had in mind has a gap, and so we will not be able to prove the most important part of the statement, namely that $e^{-\frac{1}{2}R_N}\hat{F}_H \in D$. (But it is known to be true by Schmid's SL_2-orbit theorem.)

Let us start by analyzing the effect of the various exponential factors. The following lemma says, roughly speaking, that they serve to make the filtration "compatible" with the eigenspace decomposition of R_S and H.

Lemma 13.1. *Let $S \in \operatorname{End}(V)$ be a semisimple endomorphism with real eigenvalues. For any $z \in \check{D}$, the limit*

$$\hat{z} = \lim_{x\to\infty} e^{xS}z$$

exists in $\check{D}$, and the corresponding filtration $F_{\hat{z}}$ is compatible with the eigenspace decomposition $V = \bigoplus E_\lambda(S)$. Moreover, there is a constant $C \geq 0$ such that

$$d_{\check{D}}\big(\hat{z}, e^{xS}z\big) \leq Ce^{-\delta x},$$

where $\delta > 0$ is the smallest distance between consecutive eigenvalues of S.

Proof. Since a filtration is just a collection of subspaces, it suffices to prove that for any subspace $W \subseteq V$ of dimension d, the limit

$$\hat{W} = \lim_{x\to\infty} e^{xS}V'$$

exists (in the Grassmannian of d-dimensional subspaces of V), and satisfies

$$\hat{W} = \bigoplus_{\lambda\in\mathbb{R}} \hat{W} \cap E_\lambda(S).$$

To make it clear what is going on, let us first do the case where $W = \mathbb{C}v$ is one-dimensional. Write $v = \sum_\lambda v_\lambda$, where $Sv_\lambda = \lambda v_\lambda$. Then

$$e^{xS}v = \sum_\lambda e^{\lambda x}v_\lambda.$$

Let $\mu \in \mathbb{R}$ be the largest number such that $v_\mu \neq 0$. From

$$e^{xS}(\mathbb{C}v) = \mathbb{C}\biggl(v_\mu + \sum_{\lambda<\mu} e^{-(\mu-\lambda)x}v_\lambda\biggr),$$

we see that $\lim_{x\to\infty} e^{xS}(\mathbb{C}v)$ exists and equals $\mathbb{C}v_\mu$. So the effect of the limit is to extract the component of v for the largest possible eigenvalue. Moreover, the rate of convergence is $e^{-(\mu-\mu')x}$, where $\mu' < \mu$ is the next largest eigenvalue of S.

In the general case, let $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ be the eigenvalues of S in increasing order, and consider the filtration $G_\bullet$ by increasing eigenvalues, with terms

$$G_j = E_{\lambda_1}(S) \oplus \cdots \oplus E_{\lambda_j}(S) \subseteq V.$$

Set $W_1 = W \cap G_1$; choose a subspace $W_2 \subseteq W$ such that $W_1 \oplus W_2 = W \cap G_2$; and so on. Continuing in this way, we obtain a collection of subspaces $W_1, \dots, W_n \subseteq W$, possibly zero, with the property that

$$W \cap G_j = W_1 \oplus \cdots \oplus W_j.$$

By construction, any nonzero vector $v \in W_j$ must have a nontrivial component $v_{\lambda_j} \in E_{\lambda_j}(S)$, and as we have seen above, $e^{xS}(\mathbb{C}v)$ therefore converges to $\mathbb{C}v_{\lambda_j}$ at a rate of $e^{-(\lambda_j-\lambda_{j-1})x}$. It follows that the subspace $e^{xS}W_j$ converges, at the same rate, to a subspace $\hat{W}_j \subseteq E_{\lambda_j}(S)$, which consists of the $E_{\lambda_j}(S)$-components of all the vectors in W_j. Putting everything together, we find that

$$\lim_{x\to\infty} e^{xS} W = \bigoplus_{j=1}^{n} \lim_{x\to\infty} e^{xS} W_j = \bigoplus_{j=1}^{n} \hat{W}_j.$$

The rate of convergence is $e^{-\delta x}$, where $\delta > 0$ is the smallest distance between consecutive eigenvalues of S. $\square$

Note. Here is an equivalent way for describing the limit in terms of the filtration $G_\bullet$. Projecting $W \subseteq V$ to the subquotient G_j/G_{j-1} yields the subspace

$$\frac{W \cap G_j + G_{j-1}}{G_{j-1}} \subseteq \frac{G_j}{G_{j-1}}.$$

Since $W \cap G_j + G_{j-1} = W_j + G_{j-1}$, the subspace $\hat{W}_j \subseteq E_{\lambda_j}(S)$ is exactly the preimage of the above subspace under the isomorphism $E_{\lambda_j}(S) \cong G_j/G_{j-1}$.

For the exponential factor $e^{\frac{1}{2}\log L(t) H}$, the filtration by increasing eigenvalues of H is exactly the monodromy weight filtration $W_\bullet$; moreover, the rate of convergence is $e^{-\frac{1}{2}\log L(t)} = L(t)^{-\frac{1}{2}}$, since the eigenvalues of H are typically consecutive integers. For the other exponential factor $e^{-\frac{1}{2}L(t)R_S} = e^{\frac{1}{2}L(t)(-R_S)}$, the relevant filtration is by *decreasing* eigenvalues of R_S; the rate of convergence is $e^{-\delta L(t)} = |t|^{2\delta}$, where $\delta > 0$ is the minimal distance among consecutive eigenvalues of R_S. So in both cases, what matters is the order of growth of the factor $|t|^{2\alpha} L(t)^\ell$, which grows more quickly if $\ell \in \mathbb{N}$ is larger and if $\alpha \in \mathbb{R}$ is smaller. This is consistent with the discussion in Lecture 10.

Proof of the theorem. Recall from Theorem 9.1 that Ψ extends holomorphically to the entire disk. This gives us the estimate

$$d_{\check{D}}\big(\Psi(t), \Psi(0)\big) \leq C|t|$$

for some constant $C > 0$. Here $\Psi(0) \in \check{D}$. Using Lemma 11.5, we deduce that

$$d_{\check{D}}\big(e^{-\frac{1}{2}L(t)R_S}\Psi(t), e^{-\frac{1}{2}L(t)R_S}\Psi(0)\big) \leq \|\operatorname{Ad} e^{-\frac{1}{2}L(t)R_S}\| \cdot C|t|.$$

Now the operator norm of $e^{-\frac{1}{2}L(t)R_S}$ is bounded by a constant times $|t|^{-(1-\varepsilon)}$ for some $\varepsilon > 0$, due to the fact that all eigenvalues of R_S lie in a half-open interval of length 1. Moreover, if we set

$$z_{lim} = \lim_{t\to 0} e^{-\frac{1}{2}L(t)R_S}\Psi(0) \in \check{D},$$

we also have (from Lemma 13.1) the distance estimate

$$d_{\check{D}}\big(e^{-\frac{1}{2}L(t)R_S}\Psi(0), z_{lim}\big) \leq C|t|^{2\delta},$$

where $\delta > 0$ is the smallest distance between consecutive eigenvalues of R_S. Because of the triangle inequality, we then get

$$d_{\check{D}}\big(e^{-\frac{1}{2}L(t)R_S}\Psi(t), z_{lim}\big) \leq C|t|^{\varepsilon} \tag{13.2}$$

for some constant $C > 0$ and some $\varepsilon > 0$. The filtration F_{lim} corresponding to the point $z_{lim} \in \check{D}$ can be described concretely as follows: take the filtration $F_{\Psi(0)}$, and make it compatible with the eigenspace decomposition of R_S, by projecting to the subquotients of the filtration by decreasing eigenvalues of R_S.

We can now add on the second exponential factor $e^{\frac{1}{2}\log L(t)H}$. On the one hand, Lemma 11.5 gives us

$$d_{\check{D}}\big(e^{\frac{1}{2}\log L(t)H}e^{-\frac{1}{2}L(t)R_S}\Psi(t), e^{\frac{1}{2}\log L(t)H}z_{lim}\big) \leq CL(t)^{\ell}|t|^{\varepsilon}$$

for some $\ell \in \mathbb{N}$, due to the fact that the operator norm of $e^{\frac{1}{2}\log L(t)}$ is bounded by a constant multiple of $L(t)^{\ell}$. On the other hand, the limit

$$\hat{z}_H = \lim_{t\to 0} e^{\frac{1}{2}\log L(t)H}z_{lim} \in \check{D}$$

exists, and we have the distance estimate

$$d_{\check{D}}\big(e^{\frac{1}{2}\log L(t)H}z_{lim}, \hat{z}_H\big) \leq C\cdot L(t)^{-\frac{1}{2}}.$$

Putting everything together, we find that

$$d_{\check{D}}\big(e^{\frac{1}{2}\log L(t)H}e^{-\frac{1}{2}L(t)R_S}\Psi(t), \hat{z}_H\big) \leq C\big(L(t)^{\ell}|t|^{\varepsilon} + L(t)^{-\frac{1}{2}}\big),$$

and since the right-hand side goes to zero as $t \to 0$, we conclude that the limit

$$\lim_{t\to 0} e^{\frac{1}{2}\log L(t)H}e^{-\frac{1}{2}L(t)R_S}\Psi(t) = \hat{z}_H$$

exists in $\check{D}$. As before, the filtration $\hat{F}_H$ corresponding to the point $\hat{z}_H \in \check{D}$ is obtained by starting from the filtration F_{lim}, and making it compatible with the eigenspace decomposition of H by projecting to the subquotients of the monodromy weight filtration $W_{\bullet}(R_N)$ (which is the filtration by increasing eigenvalues of H).

It is easy to see from the construction that the two semisimple operators R_S and H preserve the filtration $\hat{F}_H$. Indeed, by Lemma 13.1, the filtration F_{lim} is compatible with the eigenspace decomposition of R_S, and so $R_S F^p_{lim} \subseteq F^p_{lim}$; since R_S commutes with H, it follows that $R_S\hat{F}^p_H \subseteq \hat{F}^p_H$ for all $p \in \mathbb{Z}$. The same argument shows that also $H\hat{F}^p_H \subseteq \hat{F}^p_H$ for all $p \in \mathbb{Z}$.

The proof that $R_N\hat{F}^p_H \subseteq \hat{F}^{p-1}_H$ is a bit more involved; ultimately, it comes down to the horizontality of the period mapping $\Phi\colon \tilde{\mathbb{H}} \to D$. We start by considering the filtration corresponding to the point $\Psi(0) \in \check{D}$.

Lemma 13.3. *We have $RF^p_{\Psi(0)} \subseteq F^{p-1}_{\Psi(0)}$ for all $p \in \mathbb{Z}$.*

Proof. Recall from Lecture 12 that $\Phi(z) = g(z)\cdot o$, where $g\colon \tilde{\mathbb{H}} \to G_{\mathbb{R}}$ is a smooth function. We have seen (in Lecture 7) that the derivative of the period mapping at the point $z \in \tilde{\mathbb{H}}$ takes the tangent vector $\frac{\partial}{\partial z}$ to a horizontal tangent vector

$$\Psi_*\left(\frac{\partial}{\partial z}\right) \in F^{-1}_{\Phi(z)}\operatorname{End}(V)\Big/F^0_{\Phi(z)}\operatorname{End}(V).$$

Moreover, by general facts about homogeneous spaces (see Lecture 6), we have

$$\Psi_*\left(\frac{\partial}{\partial z}\right) = \frac{\partial}{\partial z}g(z)\cdot g(z)^{-1}.$$

Now $\Psi(e^z) = e^{-zR}\Phi(z) = e^{-zR}g(z)\cdot o$, and the image of the tangent vector $\frac{\partial}{\partial z}$ under the differential of this mapping is therefore

$$\frac{\partial}{\partial z}\Big(e^{-zR}g(z)\Big)\cdot g(z)^{-1}e^{zR} = -R + e^{-zR}\,\Psi_*\left(\frac{\partial}{\partial z}\right)e^{zR}.$$

Note that the second terms belongs to $F^{-1}_{\Psi(t)}\operatorname{End}(V)\big/F^0_{\Psi(t)}\operatorname{End}(V)$, where $t = e^z$. Since Ψ extends holomorphically over the origin, the mapping $\Psi(t) = \Psi(e^z)$ takes the tangent vector $\frac{\partial}{\partial z}$ to the vector

$$e^z\cdot\Psi_*\left(\frac{\partial}{\partial t}\right) = t\cdot\Psi_*\left(\frac{\partial}{\partial t}\right),$$

which goes to zero at a rate of $|t|$. The conclusion is that

$$R \in F^{-1}_{\Psi(0)}\operatorname{End}(V)\big/F^0_{\Psi(0)},$$

which is exactly what we wanted to show. $\square$

Next, let us prove the corresponding result for the filtration F_{lim}. Recall that

$$F_{lim} = \lim_{t\to 0} e^{-\frac{1}{2}L(t)R_S}F_{\Psi(0)}$$

is compatible with the eigenspace decomposition of R_S. Therefore

$$R_N F^p_{lim} = RF^p_{lim} = \lim_{t\to 0} e^{-\frac{1}{2}L(t)R_S)}RF^p_{\Psi(0)} \subseteq \lim_{t\to 0} e^{-\frac{1}{2}L(t)R_S)}F^{p-1}_{\Psi(0)} = F^{p-1}_{lim}.$$

Finally, we can show that $R_N\hat{F}^p_H \subseteq \hat{F}^{p-1}_H$ for all $p\in\mathbb{Z}$. By definition,

$$\hat{F}_H = \lim_{t\to 0} e^{\frac{1}{2}\log L(t)H}F_{lim}.$$

A brief computation shows that $e^{-\frac{1}{2}\log L(T)}R_N e^{\frac{1}{2}\log L(t)H} = L(t)^{-1}R_N$, and so

$$\begin{aligned} R_N\hat{F}^p_H &= \lim_{t\to 0} R_N e^{\frac{1}{2}\log L(t)H}F^p_{lim} = \lim_{t\to 0} e^{\frac{1}{2}\log L(t)H}R_N F^p_{lim} \\ &\subseteq \lim_{t\to 0} e^{\frac{1}{2}\log L(t)H}F^{p-1}_{lim} = \hat{F}^{p-1}_H. \end{aligned}$$

This is what we needed to prove.

Nilpotent orbits. The estimates from the proof of Theorem 10.2 lead to various approximations of the original period mapping $\Phi\colon\tilde{\mathbb{H}}\to D$. The first approximation is constructed from the point $\Psi(0)\in\check{D}$.

Exercise 13.1. Show that the holomorphic mapping

$$\tilde{\mathbb{H}}\to\check{D},\quad z\mapsto e^{zR}\Psi(0),$$

is horizontal, and that there is a constant $B>0$ such that $e^{zR}\Psi(0)\in D$ whenever $\operatorname{Re} z < -B$. (Hint: Use the function $g(z)$ to translate everything into a neighborhood of the base point $o\in D$.)

We can consider $e^{zR}\Psi(0)$ as the period mapping of a polarized variation of Hodge structure on a sufficiently small punctured neighborhood of the origin; the monodromy transformation is of course still $e^{2\pi iR}$. In effect, we have replaced the holomorphic mapping $\Psi\colon\tilde{\mathbb{H}}\to\check{D}$ by its value at the origin; the Hodge bundles of the new variation of Hodge structure are now *constant* subbundles of the canonical extension.

The second approximation, which Schmid's calls the *approximating nilpotent orbit*, is constructed from the point $z_{lim}\in\check{D}$.

Exercise 13.2. Show that the holomorphic mapping

$$\tilde{\mathbb{H}} \to \check{D}, \quad z \mapsto e^{zR_N} z_{lim},$$

is horizontal, and that there is a constant $B > 0$ such that $e^{zR_N} z_{lim} \in D$ whenever $\operatorname{Re} z < -B$.

The name "nilpotent orbit" comes from the fact that R_N is nilpotent. Since the filtration F_{lim} is compatible with the eigenspace decomposition of R_S, we get the same result if we replace R_N by R in the above formula. We can therefore consider $e^{zR_N} z_{lim}$ as the period mapping for a variation of Hodge structure on a sufficiently small punctured neighborhood of the origin, whose monodromy transformation is either $e^{2\pi i R_N}$ or $e^{2\pi i R}$; both choices are allowed.